

THE EHRENFEUCHT-FRAÏSSÉ-GAME OF LENGTH ω_1

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ABSTRACT. Let \mathfrak{A} and \mathfrak{B} be two first order structures of the same vocabulary. We shall consider the *Ehrenfeucht-Fraïssé-game of length ω_1 of \mathfrak{A} and \mathfrak{B}* which we denote by $\mathcal{E}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$. This game is like the ordinary Ehrenfeucht-Fraïssé-game of $L_{\omega\omega}$ except that there are ω_1 moves. It is clear that $\mathcal{E}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined if \mathfrak{A} and \mathfrak{B} are of cardinality $\leq \aleph_1$. We prove the following results:

Theorem 1. *If $V = L$, then there are models \mathfrak{A} and \mathfrak{B} of cardinality \aleph_2 such that the game $\mathcal{E}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is nondetermined.*

Theorem 2. *If it is consistent that there is a measurable cardinal, then it is consistent that $\mathcal{E}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for all \mathfrak{A} and \mathfrak{B} of cardinality $\leq \aleph_2$.*

Theorem 3. *For any $\kappa \geq \aleph_3$ there are \mathfrak{A} and \mathfrak{B} of cardinality κ such that the game $\mathcal{E}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is nondetermined.*

1. INTRODUCTION

Let \mathfrak{A} and \mathfrak{B} be two first order structures of the same vocabulary L . We denote the domains of \mathfrak{A} and \mathfrak{B} by A and B respectively. All vocabularies are assumed to be relational.

The *Ehrenfeucht-Fraïssé-game of length γ of \mathfrak{A} and \mathfrak{B}* , denoted by $\mathcal{E}_\gamma(\mathfrak{A}, \mathfrak{B})$, is defined as follows: There are two players called \forall and \exists . First \forall plays x_0 and then \exists plays y_0 . After this \forall plays x_1 , and \exists plays y_1 , and so on. If $\langle (x_\beta, y_\beta) : \beta < \alpha \rangle$ has been played and $\alpha < \gamma$, then \forall plays x_α after which \exists plays y_α . Eventually a sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ has been played. The rules of the game say that both players have to play elements of $A \cup B$. Moreover, if \forall plays his x_β in A (B), then \exists has to play his y_β in B (A). Thus the sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ determines a relation $\pi \subseteq A \times B$. Player \exists wins this round of the game if π is a partial isomorphism. Otherwise \forall wins. The notion of winning strategy is defined in the usual manner. We say that a player *wins* $\mathcal{E}_\gamma(\mathfrak{A}, \mathfrak{B})$ if he has a winning strategy in $\mathcal{E}_\gamma(\mathfrak{A}, \mathfrak{B})$.

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Recall that

$$\begin{aligned}\mathfrak{A} \equiv_{\omega\omega} \mathfrak{B} &\Leftrightarrow \forall n < \omega (\exists \text{ wins } \mathcal{G}_n(\mathfrak{A}, \mathfrak{B})) \\ \mathfrak{A} \equiv_{\infty\omega} \mathfrak{B} &\Leftrightarrow \exists \text{ wins } \mathcal{G}_\omega(\mathfrak{A}, \mathfrak{B}).\end{aligned}$$

In particular, $\mathcal{G}_\gamma(\mathfrak{A}, \mathfrak{B})$ is determined for $\gamma \leq \omega$. The question, whether $\mathcal{G}_\gamma(\mathfrak{A}, \mathfrak{B})$ is determined for $\gamma > \omega$, is the subject of this paper. We shall concentrate on the case $\gamma = \omega_1$.

The notion

$$(1) \quad \exists \text{ wins } \mathcal{G}_\gamma(\mathfrak{A}, \mathfrak{B})$$

can be viewed as a natural generalization of $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. The latter implies isomorphism for countable models. Likewise (1) implies isomorphism for models of cardinality $|\gamma|$:

Proposition 1. *Suppose \mathfrak{A} and \mathfrak{B} have cardinality $\leq \kappa$. Then $\mathcal{G}_\kappa(\mathfrak{A}, \mathfrak{B})$ is determined: \exists wins if $\mathfrak{A} \cong \mathfrak{B}$, and \forall wins if $\mathfrak{A} \not\cong \mathfrak{B}$.*

Proof. If $f: \mathfrak{A} \cong \mathfrak{B}$, then the winning strategy of \exists in $\mathcal{G}_\kappa(\mathfrak{A}, \mathfrak{B})$ is to play in such a way that the resulting π satisfies $\pi \subseteq f$. On the other hand, if $\mathfrak{A} \not\cong \mathfrak{B}$, then the winning strategy of \forall is to systematically enumerate $A \cup B$ so that the final π will satisfy $A = \text{dom}(\pi)$ and $B = \text{rng}(\pi)$. \square

For models of arbitrary cardinality we have the following simple but useful criterion of (1), namely in the terminology of [15] that they are “potentially isomorphic.” We use $\text{Col}(\lambda, \kappa)$ to denote the notion of forcing which collapses $|\lambda|$ to κ (with conditions of cardinality less than κ).

Proposition 2. *Suppose \mathfrak{A} and \mathfrak{B} have cardinality $\leq \lambda$ and κ is regular. Player \exists wins $\mathcal{G}_\kappa(\mathfrak{A}, \mathfrak{B})$ if and only if $\Vdash_{\text{Col}(\lambda, \kappa)} \mathfrak{A} \cong \mathfrak{B}$.*

Proof. Suppose τ is a winning strategy of \exists in $\mathcal{G}_\kappa(\mathfrak{A}, \mathfrak{B})$. Since $\text{Col}(\lambda, \kappa)$ is $< \kappa$ -closed,

$$\Vdash_{\text{Col}(\lambda, \kappa)} \text{“}\tau \text{ is a winning strategy of } \exists \text{ in } G_\kappa(\mathfrak{A}, \mathfrak{B}).\text{”}$$

Hence $\Vdash_{\text{Col}(\lambda, \kappa)} \mathfrak{A} \cong \mathfrak{B}$ by Proposition 1. Suppose then $p \Vdash \tilde{f}: \mathfrak{A} \cong \mathfrak{B}$ for some $p \in \text{Col}(\lambda, \kappa)$. While the game $\mathcal{G}_\kappa(\mathfrak{A}, \mathfrak{B})$ is played, \exists keeps extending the condition p further and further. Suppose he has extended p to q and \forall has played $x \in A$. Then \exists finds $r \leq q$ and $y \in B$ with $r \Vdash \tilde{f}(x) = y$. Using this simple strategy \exists wins. \square

Proposition 3. *Suppose T is an ω -stable first order theory with NDOP. Then $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for all models \mathfrak{A} of T and all models \mathfrak{B} .*

Proof. Suppose \mathfrak{A} is a model of T . If \mathfrak{B} is not $L_{\infty\omega_1}$ -equivalent to \mathfrak{A} , then \forall wins $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ easily. So let us suppose $\mathfrak{A} \equiv_{\infty\omega_1} \mathfrak{B}$. We may assume A and B are of cardinality $\geq \aleph_1$. If we collapse $|A|$ and $|B|$ to \aleph_1 , T will remain ω -stable with NDOP, and \mathfrak{A} and \mathfrak{B} will remain $L_{\infty\omega_1}$ -equivalent. So \mathfrak{A} and \mathfrak{B} become isomorphic by [19, Chapter XIII, §1]. Now Proposition 2 implies that \exists wins $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$. \square

Hyttinen [10] showed that $\mathcal{G}_\gamma(\mathfrak{A}, \mathfrak{B})$ may be nondetermined for all γ with $\omega < \gamma < \omega_1$ and asked whether $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ may be nondetermined. Our results

show that $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ may be nondetermined for \mathfrak{A} and \mathfrak{B} of cardinality \aleph_3 (Theorem 17), but for models of cardinality \aleph_2 the answer is more complicated.

Let $F(\omega_1)$ be the free group of cardinality \aleph_1 . Using the combinatorial principle \square_{ω_1} we construct an abelian group G of cardinality \aleph_2 such that $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is nondetermined (Theorem 4). On the other hand, we show that starting with a model with a measurable cardinal one can build a forcing extension in which $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for all models \mathfrak{A} and \mathfrak{B} of cardinality $\leq \aleph_2$ (Theorem 14).

Thus the free abelian group $F(\omega_1)$ has the remarkable property that the question

Is $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ determined for all G ?

cannot be answered in ZFC alone. Proposition 3 shows that no model of an \aleph_1 -categorical first order theory can have this property.

We follow Jech [11] in set theoretic notation. We use S_n^m to denote the set $\{\alpha < \omega_m : \text{cf}(\alpha) = \omega_n\}$. Closed and unbounded sets are called cub sets. A set of ordinals is λ -closed if it is closed under supremums of ascending λ -sequences $\langle \alpha_i : i < \lambda \rangle$ of its elements. A subset of a cardinal is λ -stationary if it meets every λ -closed unbounded subset of the cardinal. The closure of a set A of ordinals in the order topology of ordinals is denoted by \overline{A} . The free abelian group of cardinality κ is denoted by $F(\kappa)$.

2. A NONDETERMINED $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ WITH G A GROUP OF CARDINALITY \aleph_2

In this section we use \square_{ω_1} to construct a group G of cardinality \aleph_2 such that the game $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is nondetermined (Theorem 4). For background on almost free groups the reader is referred to [4]. However, our presentation does not depend on special knowledge of almost free groups. All groups below are assumed to be abelian.

By \square_{ω_1} we mean the principle, which says that there is a sequence $\langle C_\alpha : \alpha < \omega_2, \alpha = \bigcup \alpha \rangle$ such that

1. C_α is a cub subset of α .
2. If $\text{cf}(\alpha) = \omega$, then $|C_\alpha| = \omega$.
3. If γ is a limit point of C_α , then $C_\gamma = C_\alpha \cap \gamma$.

Recall that \square_{ω_1} follows from $V = L$ by a result of R. Jensen [14]. For a sequence of sets C_α as above we can let

$$E_\beta = \{\alpha \in S_0^2 : \text{the order type of } C_\alpha \text{ is } \beta\}.$$

For some $\beta < \omega_1$ the set E_β has to be stationary. Let us use E to denote this E_β . Then E is a so-called *nonreflecting* stationary set, i.e., if $\text{cf}(\gamma) > \omega$ then $E \cap \gamma$ is nonstationary on γ . Indeed, then some final segment D_γ of the set of limit points of C_γ is a cub subset of γ disjoint from E . Moreover, $\text{cf}(\alpha) = \omega$ for all $\alpha \in E$.

Theorem 4. *Assuming \square_{ω_1} , there is a group G of cardinality \aleph_2 such that the game $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is nondetermined.*

Proof. Let \mathbb{Z}^{ω_2} denote the direct product of ω_2 copies of the additive group \mathbb{Z} of the integers. Let x_α be the element of \mathbb{Z}^{ω_2} which is 0 on coordinates $\neq \alpha$ and 1 on the coordinate α . Let us fix for each $\delta \in S_0^2$ an ascending cofinal

sequence $\eta_\delta: \omega \rightarrow \delta$. For such δ , let

$$z_\delta = \sum_{n=0}^{\infty} 2^n x_{\eta_\delta(n)}.$$

Let $\langle C_\alpha: \alpha = \bigcup \alpha < \omega_2 \rangle$, $\langle D_\alpha: \alpha = \bigcup \alpha < \omega_2 \rangle$, and $E = E_\beta$ be obtained from \square_{ω_1} as above. We are ready to define the groups we need for the proof: Let G be the smallest pure subgroup of \mathbb{Z}^{ω_2} which contains x_α for $\alpha < \omega_2$ and z_δ for $\delta \in E$, let G_α be the smallest pure subgroup of \mathbb{Z}^{ω_2} which contains x_γ for $\gamma < \alpha$ and z_δ for $\delta \in E \cap \alpha$, let $F (= F(\omega_2))$ be the subgroup of \mathbb{Z}^{ω_2} generated freely by x_α for $\alpha < \omega_2$, and finally, let F_α be the subgroup of \mathbb{Z}^{ω_2} generated freely by x_γ for $\gamma < \alpha$.

The properties we shall want of G_α are standard but for the sake of completeness we shall sketch proofs. We need that each G_α is free and for any $\beta \notin E$ any free basis of G_β can be extended to a free basis of G_α for all $\alpha > \beta$.

The proof is by induction on α . For limit ordinals we use the fact that E is nonreflecting. The case of successors of ordinals not in E is also easy. Assume now that $\delta \in E$ and the induction hypothesis has been verified up to δ . By the induction hypothesis for any $\beta < \delta$ such that $\beta \notin E$, there is n_0 so that $G_\delta = G_\beta \oplus H \oplus K$ where K is the group freely generated by $\{x_{\eta_\delta(n)}: n_0 \leq n\}$ and $x_{\eta_\delta(m)} \in G_\beta$ for all $m < n_0$. Then $G_{\delta+1} = G_\beta \oplus H \oplus K'$ where K' is freely generated by $\{\sum_{m=n}^{\infty} 2^{m-n} x_{\eta_\delta(m)}: n_0 \leq n\}$.

On the other hand, if $\delta \in E$ and $\{x_{\eta_\delta(n)}: n < \omega\} \subseteq B$, where B is a subgroup of G such that $z_\delta \notin B$, then G/B is nonfree, as $z_\delta + B$ is infinitely divisible by 2 in G/B .

Claim 1. \exists does not win $\mathcal{G}_{\omega_1}(F, G)$.

Suppose τ is a winning strategy of \exists . Let $\alpha \in E$ such that the pair (G_α, F_α) is closed under the first ω moves of τ , that is, if \forall plays his first ω moves inside $G_\alpha \cup F_\alpha$, then τ orders \exists to do the same. We shall play $\mathcal{G}_{\omega_1}(F, A)$ pointing out the moves of \forall and letting τ determine the moves of \exists . On his move number $2n$ \forall plays the element $x_{\eta_\alpha(n)}$ of G_α . On his move number $2n+1$ \forall plays some element of F_α . Player \forall plays his moves in F_α in such a way that during the first ω moves eventually some countable direct summand K of F_α as well as some countable $B \subseteq G_\alpha$ are enumerated. Let J be the smallest pure subgroup of G containing $B \cup \{z_\alpha\}$. During the next ω moves of $\mathcal{G}_{\omega_1}(F, A)$ player \forall enumerates J and \exists responds by enumerating some $H \subseteq F$. Since τ is a winning strategy, H has to be a subgroup of F . But now H/K is free, whereas J/B is nonfree, so \forall will win the game, a contradiction.

Claim 2. \forall does not win $\mathcal{G}_{\omega_1}(F, G)$.

Suppose τ is a winning strategy of \forall . If we were willing to use CH, we could just take α of cofinality ω_1 such that (F_α, G_α) is closed under τ , and derive a contradiction from the fact that $F_\alpha \cong G_\alpha$. However, since we do not want to assume CH, we have to appeal to a longer argument.

Let $\kappa = (2^\omega)^{++}$. Let \mathfrak{M} be the expansion of $\langle H(\kappa), \in \rangle$ obtained by adding the following structure to it:

(H1) The function $\delta \mapsto \eta_\delta$.

(H2) The function $\delta \mapsto z_\delta$.

(H3) The function $\alpha \mapsto C_\alpha$.

(H4) A well-ordering $<$ of the universe.

(H5) The winning strategy τ .

Let $\mathfrak{N} = \langle N, \in, \dots \rangle$ be an elementary submodel of \mathfrak{M} such that $\omega_1 \subseteq N$ and $N \cap \omega_2$ is an ordinal α of cofinality ω_1 .

Let $D_\alpha = \{\beta_i : i < \omega_1\}$ in ascending order. Since $C_{\beta_i} = C_\alpha \cap \beta_i$, every initial segment of C_α is in N . By elementarity, $G_{\beta_i} \in N$ for all $i < \omega_1$. Let ϕ be an isomorphism $G_\alpha \rightarrow F_\alpha$ obtained as follows: ϕ restricted to G_{β_0} is the $<$ -least isomorphism between the free groups G_{β_0} and F_0 . If ϕ is defined on all G_{β_j} , $j < i$, then ϕ is defined on G_{β_i} as the $<$ -least extension of $\bigcup_{j < i} \phi_{\beta_j}$ to an isomorphism between G_{β_i} and F_i . Recall that by our choice of D_α , $G_{\beta_{i+1}}/G_{\beta_i}$ is free, so such extensions really exist.

We derive a contradiction by showing that \exists can play ϕ against τ for the whole duration of the game $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$. To achieve this we have to show that when \exists plays his canonical strategy based on ϕ , the strategy τ of \forall directs \forall to go on playing elements which are in N , that is, elements of $G_\alpha \cup F_\alpha$.

Suppose a sequence $s = \langle (x_\gamma, y_\gamma) : \gamma < \mu \rangle$, $\mu < \omega_1$, has been played. It suffices to show that $s \in N$. Choose β_i so that the elements of s are in $G_{\beta_i} \cup F_{\beta_i}$. Now s is uniquely determined by $\phi \upharpoonright G_{\beta_i}$ and τ . Note that because $C_{\beta_i} = C_\alpha \cap \beta_i$, $\phi \upharpoonright G_{\beta_i}$ can be defined inside N similarly as ϕ was defined above, using C_{β_i} instead of C_α . Thus $s \in N$ and we are done.

We have proved that $\mathcal{G}_{\omega_1}(F, G)$ is nondetermined. This clearly implies $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is nondetermined. \square

Remark. R. Jensen [14, p. 286] showed that if \square_{ω_1} fails, then ω_2 is Mahlo in L . Therefore, if $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for all almost free groups \mathfrak{A} and \mathfrak{B} of cardinality \aleph_2 , then ω_2 is Mahlo in L . If we start with \square_κ , we get an almost free group A of cardinality κ^+ such that $\mathcal{G}_{\omega_1}(F(\omega_1), A)$ is nondetermined.

3. $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ CAN BE DETERMINED FOR ALL G

In this section all groups are assumed to be abelian. It is easy to see that \exists wins $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ for any uncountable free group G , so in this exposition $F(\omega_1)$ is a suitable representative of all free groups. In the study of determinacy of $\mathcal{G}_{\omega_1}(F(\omega_1), \mathfrak{A})$ for various \mathfrak{A} it suffices to study groups \mathfrak{A} , since for other \mathfrak{A} player \forall easily wins the game.

Starting from a model with a Mahlo cardinal we construct a forcing extension in which $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is determined, when G is any group of cardinality \aleph_2 . This can be extended to groups G of any cardinality, if we start with a supercompact cardinal.

In the proof of the next results we shall make use of *stationary logic* $L(aa)$ introduced in [17]. For the definition and basic facts about $L(aa)$ the reader is referred to [1]. This logic has a new quantifier $aa s$ quantifying over variables s ranging over countable subsets of the universe. A cub set of such s is any set which contains a superset of any countable subset of the universe and which is closed under unions of countable chains. The semantics of $aa s$ is defined as follows:

$$aa s \phi(s, \dots) \Leftrightarrow \phi(s, \dots) \text{ holds for a cub set of } s.$$

Note that a group of cardinality \aleph_1 is free if and only if it satisfies

(+) $aa \ s \ aa \ s' (s \subseteq s' \rightarrow s \text{ is free and } s'/s \text{ is free}).$

Proposition 5. *Let G be a group of cardinality at most \aleph_2 . Then the following conditions are equivalent:*

- (1) \exists wins $\mathcal{G}_{\omega_1}(F(\omega_1), G)$.
- (2) G satisfies (+).
- (3) G is the union of a continuous chain $\langle G_\alpha : \alpha < \omega_2 \rangle$ of free subgroups with $G_{\alpha+1}/G_\alpha$ \aleph_1 -free for all $\alpha < \omega_2$.

Proof. (1) implies (2): Suppose \exists wins $\mathcal{G}_{\omega_1}(F(\omega_1), G)$. By Proposition 2 we have $\Vdash_{\text{Col}(|G|, \omega_1)} "G \text{ is free}"$. Using the countable completeness of $\text{Col}(|G|, \omega_1)$ it is now easy to construct a cub set S of countable subgroups of G such that if $A \in S$ then for all $B \in S$ with $A \subseteq B$ we have B/A free. Thus G satisfies (+). (2) implies (3) quite trivially. (3) implies (1): Suppose a continuous chain as in (3) exists. If we collapse $|G|$ to \aleph_1 , then in the extension the chain has length $< \omega_2$. Now we use Theorem 1 of [8]:

If a group A is the union of a continuous chain of $< \omega_2$ free subgroups $\{A_\alpha : \alpha < \gamma\}$ of cardinality $\leq \aleph_1$ such that each $A_{\alpha+1}/A_\alpha$ is \aleph_1 -free, then A is free.

Thus G is free in the extension and (1) follows from Proposition 2. \square

Remark. Conditions (1) and (2) of Proposition 5 are equivalent for G of any size.

Let us consider the following principle:

- (*) For all stationary $E \subseteq S_0^2$ and countable subsets a_α of $\alpha \in E$ such that a_α is cofinal in α and of order type ω there is a closed $C \subseteq \omega_2$ of order type ω_1 such that $\{\alpha \in E : a_\alpha \setminus C \text{ is finite}\}$ is stationary in C .

Lemma 6. *The principle (*) implies that $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is determined for all groups G of cardinality \aleph_2 .*

Proof. Suppose G is a group of cardinality \aleph_2 . We may assume the domain of G is ω_2 . Let us assume G is \aleph_2 -free, as otherwise \forall easily wins. We prove that G satisfies condition (3) of Proposition 5 and thereby that \exists wins $\mathcal{G}_{\omega_1}(F(\omega_1), G)$.

To prove condition (3) of Proposition 5, assume the contrary. Then G can be expressed as the union of a continuous chain $\langle G_\alpha : \alpha < \omega_2 \rangle$ of free groups with $G_{\alpha+1}/G_\alpha$ non- \aleph_1 -free for $\alpha \in E$, $E \subseteq \omega_2$ stationary. By Fodor's Lemma, we may assume $E \subseteq S_0^2$ or $E \subseteq S_1^2$. The latter case is much easier and therefore we assume $E \subseteq S_0^2$. Also we may assume that for all α , every ordinal in $G_{\alpha+1} \setminus G_\alpha$ is greater than every ordinal in G_α . Finally by intersecting with a closed unbounded set we may assume that for all $\alpha \in E$ the set underlying G_α is α . Choose for each $\alpha \in E$ some countable subgroup b_α of $G_{\alpha+1}$ with $b_\alpha + G_\alpha/G_\alpha$ nonfree. Let $c_\alpha = b_\alpha \cap G_\alpha$. We will choose a_α so that any final segment generates a subgroup containing c_α . Enumerate c_α as $\{g_n : n < \omega\}$ such that each element is enumerated infinitely often. Choose an increasing sequence $(\alpha_n : n < \omega)$ cofinal in α so that for all n , $g_n \in G_{\alpha_n}$. Finally, for each n , choose $h_n \in G_{\alpha_{n+1}} \setminus G_{\alpha_n}$. Let $a_\alpha = \{h_n : n < \omega\} \cup \{h_n + g_n : n < \omega\}$. It

is now easy to check that a_α is a sequence of order type ω which is cofinal in α and any subgroup of G which contains all but finitely many of the elements of a_α contains c_α .

By (*) there is a continuous C of order type ω_1 such that $\{\alpha \in C: a_\alpha \setminus C \text{ is finite}\}$ is stationary in C . Let $D = \langle C \cup \sum_{\alpha \in C} b_\alpha \rangle$. Since $|D| \leq \aleph_1$, D is free.

For any $\alpha \in C$, let

$$D_\alpha = \left\langle (C \cap \alpha) \cup \left(\sum_{\beta \in (C \cap \alpha)} b_\beta \right) \right\rangle.$$

Note that $D = \bigcup_{\alpha \in C} D_\alpha$, each D_α is countable and for limit point δ of C , $D_\delta = \bigcup_{\alpha \in (C \cap \delta)} D_\alpha$. Hence there is an $\alpha \in C \cap E$ such that $a_\alpha \setminus C$ is finite and D/D_α is free. Hence $b_\alpha + D_\alpha/D_\alpha$ is free. But $b_\alpha + D_\alpha/D_\alpha \cong b_\alpha/b_\alpha \cap D_\alpha = b_\alpha/b_\alpha \cap G_\alpha$, which is not free, a contradiction. \square

For the next theorem we need a lemma from [6]. A proof is included for the convenience of the reader.

Lemma 7 [6]. *Suppose λ is a regular cardinal and \mathbb{Q} is a notion of forcing which satisfies the λ -c.c. Suppose \mathcal{I} is a normal λ -complete ideal on λ and $\mathcal{I}^+ = \{S \subseteq \lambda: S \notin \mathcal{I}\}$. For all sets $S \in \mathcal{I}^+$ and sequences of conditions $\langle p_\alpha: \alpha \in S \rangle$, there is a set C with $\lambda \setminus C \in \mathcal{I}$ so that for all $\alpha \in C \cap S$,*

$$p_\alpha \Vdash_{\mathbb{Q}} \{ \beta: p_\beta \in \tilde{G} \} \in \mathcal{I}^+, \text{ where } \mathcal{I} \text{ is the ideal generated by } \mathcal{I}^+.$$

Proof. Suppose the lemma is false. So there is an \mathcal{I} -positive set $S' \subseteq S$ such that for all $\alpha \in S'$ there is an extension r_α of p_α and a set $I_\alpha \in \mathcal{I}$ (note: I_α is in the ground model) so that

$$r_\alpha \Vdash \{ \beta: p_\beta \in \tilde{G} \} \subseteq I_\alpha.$$

Let I be the diagonal union of $\{I_\alpha: \alpha \in S'\}$.

Suppose now that $\alpha < \beta$ and $\alpha, \beta \in (S' \setminus I)$. Since $\beta \notin I$, $r_\alpha \Vdash p_\beta \notin \tilde{G}$. Hence $r_\alpha \Vdash r_\beta \notin \tilde{G}$. So r_α, r_β are incompatible. Hence $\{r_\alpha: \alpha \in S' \setminus I\}$ is an antichain which, since S' is \mathcal{I} -positive, is of cardinality λ . This is a contradiction. \square

Remark. It is a well-known fact that the ideal \mathcal{I} of Lemma 7 is forced by \mathbb{Q} to be λ -complete and normal.

Theorem 8. *Assuming the consistency of a Mahlo cardinal, it is consistent that (*) holds and hence that $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is determined for all groups G of cardinality \aleph_2 .*

Proof. By a result of Harrington and Shelah [7] we may start with a Mahlo cardinal κ in which every stationary set of cofinality ω reflects, that is, if $S \subseteq \kappa$ is stationary, and $\text{cf}(\alpha) = \omega$ for $\alpha \in S$, then $S \cap \lambda$ is stationary in λ for some inaccessible $\lambda < \kappa$.

For any inaccessible λ let \mathbb{P}_λ be the Levy-forcing for collapsing λ to ω_2 . The conditions of \mathbb{P}_λ are countable functions $f: \lambda \times \omega_1 \rightarrow \lambda$ such that $f(\alpha, \beta) < \alpha$ for all α and β and each f is increasing and continuous in the second coordinate. It is well known that \mathbb{P}_λ is countably closed and satisfies the λ -chain condition [11, p. 191].

Let $\mathbb{P} = \mathbb{P}_\kappa$. Suppose $p \in \mathbb{P}$ and

$$p \Vdash \begin{array}{l} \text{"}\tilde{E} \subseteq S_0^2 \text{ is stationary and} \\ \forall \alpha \in \tilde{E} (\tilde{a}_\alpha \subseteq \alpha \text{ is cofinal in } \alpha \text{ and of order type } \omega)\text{"} \end{array}$$

Let

$$S = \{\alpha < \kappa : \exists q \leq p (q \Vdash \alpha \in \tilde{E})\}.$$

For any $\alpha \in S$ let $p_\alpha \leq p$ such that $p_\alpha \Vdash \alpha \in \tilde{E}$. Since \mathbb{P} is countably closed, we can additionally require that for some countable $a_\alpha \subseteq \alpha$ we have $p_\alpha \Vdash \tilde{a}_\alpha = a_\alpha$.

The set S is stationary in κ , for if $C \subseteq \kappa$ is cub, then $p \Vdash C \cap \tilde{E} \neq \emptyset$, whence $C \cap S \neq \emptyset$. Also $\text{cf}(\alpha) = \omega$ for $\alpha \in S$. Let λ be inaccessible such that $S \cap \lambda$ is stationary in λ . We may choose λ in such a way that $\alpha \in S \cap \lambda$ implies $p_\alpha \in \mathbb{P}_\lambda$. By Lemma 7 there is a $\delta \in S \cap \lambda$ such that

$$p_\delta \Vdash_{\mathbb{P}_\lambda} \text{"}\tilde{E}_1 = \{\alpha < \lambda : p_\alpha \in \tilde{G}\} \text{ is stationary.}"$$

Let \mathbb{Q} be the set of conditions $f \in \mathbb{P}$ with $\text{dom}(f) \subseteq (\kappa \setminus \lambda) \times \omega_1$. Note that $\mathbb{P} \cong \mathbb{P}_\lambda \otimes \mathbb{Q}$. Let G be \mathbb{P} -generic containing p_δ and $G_\lambda = G \cap \mathbb{P}_\lambda$ for any inaccessible $\lambda \leq \kappa$. Then G_κ is \mathbb{P}_λ -generic and ω_2 of $V[G_\lambda]$ is λ . Let us work now in $V[G_\lambda]$. Thus λ is the current ω_2 , $E_1 = \{\alpha < \lambda : p_\alpha \in G_\lambda\}$ is stationary, and we have the countable sets $a_\alpha \subseteq \alpha$ for $\alpha \in E_1$. Since \mathbb{Q} collapses λ there is a name \tilde{f} such that

$$\Vdash_{\mathbb{Q}} \text{"}\tilde{f} : \omega_1 \rightarrow \lambda \text{ is continuous and cofinal.}"$$

More precisely \tilde{f} is the name for the function f defined by $f(\alpha) = \beta$ if and only if there is some $g \in G$ so that $g(\lambda, \alpha) = \beta$. Let \tilde{C} denote the range of \tilde{f} . We shall prove the following statement:

Claim. $\Vdash_{\mathbb{Q}} \{\alpha < \lambda : a_\alpha \setminus \tilde{C} \text{ is finite}\}$ is stationary in \tilde{C} .

Suppose $q \in \mathbb{Q}$ so that $q \Vdash \text{"}\tilde{D} \subseteq \omega_1 \text{ is a cub.}"$ Let \mathfrak{M} be an appropriate expansion of $\langle H(\kappa), \in \rangle$ and $\langle \mathfrak{N}_i : i < \lambda \rangle$, $\mathfrak{N}_i = \langle N_i, \in, \dots \rangle$, a sequence of elementary submodels of \mathfrak{M} such that

- (i) Everything relevant is in N_0 .
- (ii) If $\alpha_i = N_i \cap \lambda$, then $\alpha_i < \alpha_j$ for $i < j < \lambda$.
- (iii) N_{i+1} is closed under countable sequences.
- (iv) $|N_i| = \omega_1$.
- (v) $N_i = \bigcup_{j < i} N_j$ for i a limit ordinal.

Choose $\gamma = \alpha_i \in E_1$ and let $\langle i_n : n < \omega \rangle$ be a sequence of successor ordinals such that $\gamma = \sup\{\alpha_{i_n} : n < \omega\}$. Let $q_0 \leq q$ and $\beta_0 \in \omega_1$ such that $q_0, \beta_0 \in N_{i_0}$,

$$q_0 \Vdash \text{"}\beta_0 \in \tilde{D}\text{"}$$

and q_0 decides the value of $\tilde{f}''\beta_0$ (which will by elementarity necessarily be a subset of α_{i_0}).

If q_n and β_n are defined we choose $q_{n+1} \leq q_n$ and $\beta_{n+1} \in \omega_1$ such that $q_{n+1}, \beta_{n+1} \in N_{i_{n+1}}$,

$$q_{n+1} \Vdash \text{"}\beta_{n+1} \in \tilde{D} \text{ and } a_\gamma \cap (\alpha_{i_{n+1}} \setminus \alpha_{i_n}) \subseteq \tilde{f}''\beta_{n+1} \subseteq \alpha_{i_{n+1}}\text{"}$$

and q_{n+1} decides $\tilde{f}''\beta_{n+1}$. Finally, let

$$q_\omega = \bigcup \{q_n : n < \omega\} \quad \text{and} \quad \beta = \bigcup \{\beta_n : n < \omega\}.$$

Then

$$q_\omega \Vdash \text{“}\beta \in \tilde{D} \quad \text{and} \quad a_\gamma \setminus \tilde{f}''\beta \text{ is finite.”}$$

The claim, and thereby the theorem, is proved. \square

Corollary 9. *The statement that $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for every structure \mathfrak{A} of cardinality \aleph_2 and every uncountable free group \mathfrak{B} , is equiconsistent with the existence of a Mahlo cardinal.*

Remark. If $\mathcal{G}_{\omega_1}(A, F(\omega_1))$ is determined for all groups A of cardinality κ^+ , κ singular, then \square_κ fails. This implies that the Covering Lemma fails for the Core Model, whence there is an inner model for a measurable cardinal. This shows that the conclusion of Theorem 8 cannot be strengthened to arbitrary G . However, by starting with a larger cardinal we can make this extension:

Theorem 10. *Assuming the consistency of a supercompact cardinal, it is consistent that $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is determined for all groups G .*

Proof. Let us assume that the stationary logic $L_{\omega_1\omega}(aa)$ has the Löwenheim-Skolem property down to \aleph_1 . This assumption is consistent relative to the consistency of a supercompact cardinal [2]. Let G be an arbitrary \aleph_2 -free group. Let H be an $L(aa)$ -elementary submodel of G of cardinality \aleph_1 . Thus H is a free group. The group H satisfies the sentence $(+)$, whence so does G . Now the claim follows from Proposition 5. \square

Corollary 11. *Assuming the consistency of a supercompact cardinal, it is consistent that $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for every structure \mathfrak{A} and every uncountable free group \mathfrak{B} .*

4. $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ CAN BE DETERMINED FOR ALL \mathfrak{A} AND \mathfrak{B} OF CARDINALITY \aleph_2

We prove the consistency of the statement that $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for all \mathfrak{A} and \mathfrak{B} of cardinality $\leq \aleph_2$ assuming the consistency of a measurable cardinal. Actually we make use of an assumption that we call $I^*(\omega)$ concerning stationary subsets of ω_2 . This assumption is known to imply that ω_2 is measurable in an inner model. It follows from the previous section that some large cardinal axioms are needed to prove the stated determinacy.

Let $I^*(\omega)$ be the following assumption about ω_1 -stationary subsets of ω_2 :

$I^*(\omega)$: Let \mathcal{I} be the ω_1 -nonstationary ideal NS_{ω_1} on ω_2 . Then \mathcal{I}^+ has a σ -closed dense subset K .

Hodges and Shelah [9] define a principle $I(\omega)$, which is like $I^*(\omega)$ except that \mathcal{I} is not assumed to be the ω_1 -nonstationary ideal. They use $I(\omega)$ to prove the determinacy of an Ehrenfeucht-Fraïssé-game played on several boards simultaneously.

Note that $I^*(\omega)$ implies \mathcal{I} is precipitous, so the consistency of $I^*(\omega)$ implies the consistency of a measurable cardinal [12]. For the proof of the following result the reader is referred to [12]:

Theorem 12 (Laver). *The assumption $I^*(\omega)$ is consistent relative to the consistency of a measurable cardinal.*

We shall consider models \mathfrak{A} , \mathfrak{B} of cardinality \aleph_2 , so we may as well assume they have ω_2 as universe. For such \mathfrak{A} and $\alpha < \omega_2$ we let \mathfrak{A}_α denote the structure $\mathfrak{A} \restriction \alpha$. Similarly \mathfrak{B}_α .

Lemma 13. *Suppose \mathfrak{A} and \mathfrak{B} are structures of cardinality \aleph_2 . If \forall does not have a winning strategy in $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$, then $S = \{\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha\}$ is ω_1 -stationary.*

Proof. Let $C \subseteq \omega_2$ be ω_1 -closed and unbounded. Suppose $S \cap C = \emptyset$. We derive a contradiction by describing a winning strategy of \forall : Let $\pi : \omega_1 \times \omega_1 \times 2 \rightarrow \omega_1$ be onto with $\alpha, \beta, d \leq \pi(\alpha, \beta, d)$ for all $\alpha, \beta < \omega_1$ and $d < 2$. If $\alpha < \omega_2$, let $\theta_\alpha : \omega_1 \rightarrow \alpha$ be onto. Suppose the sequence $\langle (x_i, y_i) : i < \alpha \rangle$ has been played. Here x_i denotes a move of \forall and y_i a move of \exists . During the game \forall has built an ascending sequence $\{c_i : i < \alpha\}$ of elements of C . Now he lets c_α be the smallest element of C greater than all the elements x_i, y_i , $i < \alpha$. Suppose $\alpha = \pi(i, \gamma, d)$. Now \forall will play $\theta_{c_i}(\gamma)$ as an element of \mathfrak{A} , if $d = 0$, and as an element of \mathfrak{B} if $d = 1$.

After all ω_1 moves of $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ have been played, some \mathfrak{A}_α and \mathfrak{B}_α , where $\alpha \in C$, have been enumerated. Since $\alpha \notin S$, \forall has won the game. \square

Theorem 14. *Assume $I^*(\omega)$. The game $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for all \mathfrak{A} and \mathfrak{B} of cardinality $\leq \aleph_2$.*

Proof. Suppose \forall does not have a winning strategy. By Lemma 13 the set $S = \{\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha\}$ is ω_1 -stationary. Let I and K be as in $I^*(\omega)$. If $\alpha \in S$, let $h_\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha$. We describe a winning strategy of \exists . The idea of this strategy is that \exists lets the isomorphisms h_α determine his moves. Of course, different h_α may give different information to \exists , so he has to decide which h_α to follow. The key point is that \exists lets some h_α determine his move only if there are stationarily many other h_β that agree with h_α on this move.

Suppose the sequence $\langle (x_i, y_i) : i < \alpha \rangle$ has been played. Again x_i denotes a move of \forall and y_i a move of \exists . Suppose \forall plays next x_α and this is (say) in A . During the game \exists has built a descending sequence $\{S_i : i < \alpha\}$ of elements of K with $S_0 \subseteq S$. The point of the sets S_i is that \exists has taken care that for all $i < \alpha$ and $\beta \in S_i$ we have $y_i = h_\beta(x_i)$ or $x_i = h_\beta(y_i)$ depending on whether \forall played x_i in A or B . Now \exists lets $S'_\alpha \subseteq \bigcap_{i < \alpha} S_i$ so that $S'_\alpha \in K$ and $\forall i \in S'_\alpha$ ($x_\alpha < i$). For each $i \in S'_\alpha$ we have $h_i(x_\alpha) < i$. By normality, there are an $S_\alpha \subseteq S'_\alpha$ in K and a y_α such that $h_i(x_\alpha) = y_\alpha$ for all $i \in S_\alpha$. This element y_α is the next move of \exists . Using this strategy \exists wins. \square

5. A NONDETERMINED $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ WITH \mathfrak{A} AND \mathfrak{B} OF CARDINALITY \aleph_3

We construct directly in ZFC two models \mathfrak{A} and \mathfrak{B} of cardinality \aleph_3 with $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ nondetermined. It readily follows that such models exist in all cardinalities $\geq \aleph_3$. The construction uses a square-like principle (Lemma 16), which is provable in ZFC.

Lemma 15 [18]. *There is a stationary $X \subseteq S_1^3$ and a sequence $\langle D_\alpha : \alpha \in X \rangle$ such that*

1. D_α is a cub subset of α for all $\alpha \in X$.

2. The order type of D_α is ω_1 .
 3. If $\alpha, \beta \in X$ and $\gamma < \min\{\alpha, \beta\}$ is a limit of both D_α and D_β , then $D_\alpha \cap \gamma = D_\beta \cap \gamma$.
 4. If $\gamma \in D_\alpha$, then γ is a limit point of D_α if and only if γ is a limit ordinal.
- Proof.* See [18, Theorem 4.1] or [3, Lemma 7.7]. \square

Lemma 16. *There are sets S, T , and C_α for $\alpha \in S$ such that the following hold:*

1. $S \subseteq S_0^3 \cup S_1^3$ and $S \cap S_1^3$ is stationary.
2. $T \subseteq S_0^3$ is stationary and $S \cap T = \emptyset$.
3. If $\alpha \in S$, then $C_\alpha \subseteq \alpha \cap S$ is closed and of order-type $\leq \omega_1$.
4. If $\alpha \in S$ and $\beta \in C_\alpha$, then $C_\beta = C_\alpha \cap \beta$.
5. If $\alpha \in S \cap S_1^3$, then C_α is cub on α .

Proof. Let X and $\langle D_\alpha : \alpha \in X \rangle$ be as in Lemma 15. Let $S' = X \cup Y$, where Y consists of ordinals which are limit points $< \alpha$ of some D_α , $\alpha \in X$. If $\alpha \in X$, we let C_α be the set of limit points $< \alpha$ of D_α . If $\alpha \in Y$, we let C_α be the set of limit points $< \alpha$ of $D_\beta \cap \alpha$, where $\beta > \alpha$ is chosen arbitrarily from X .

Now claims 1, 3, 4 and 6 are clearly satisfied.

Let $S_0^3 = \bigcup_{i < \omega_2} T_i$ where the T_i are disjoint stationary sets. Since $|\overline{C_\alpha}| \leq \omega_1$, there is $i_\alpha < \omega_2$ such that $i \geq i_\alpha$ implies $\overline{C_\alpha} \cap T_i = \emptyset$. Let $S'' \subseteq S'$ be stationary such that $\alpha \in S''$ implies i_α is constant i . Let $T = T_i$. Finally, let $S = S'' \cup \bigcup \{C_\alpha : \alpha \in S''\}$. Claim 2 is satisfied, and the lemma is proved. \square

Theorem 17. *There are structures \mathfrak{A} and \mathfrak{B} of cardinality \aleph_3 with one binary predicate such that the game $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is nondetermined.*

Proof. Let S, T , and $\langle C_\alpha : \alpha \in S \rangle$ be as in Lemma 16. We shall construct a sequence $\{M_\alpha : \alpha < \omega_3\}$ of sets and a sequence $\{G_\alpha : \alpha \in S\}$ of functions such that the conditions (M1)–(M5) below hold. Let W_α be the set of all mappings $G_{\gamma_0}^{d_0} \dots G_{\gamma_n}^{d_n}$, where $\gamma_0, \dots, \gamma_n \in S \cap \alpha$, $d_i \in \{-1, 1\}$, G_γ^1 means G_γ and G_γ^{-1} means the inverse of G_γ . Let $W = W_{\omega_3}$. (Note that W consists of a set of partial functions.)

The conditions on the M_α 's and the G_α 's are

- (M1) $M_\alpha \subseteq M_\beta$ if $\alpha < \beta$, and $M_\alpha \subset M_{\alpha+1}$ if $\alpha \in S$.
- (M2) $M_\nu = \bigcup_{\alpha < \nu} M_\alpha$ for limit ν .
- (M3) G_α is a bijection of $M_{\alpha+1}$ for $\alpha \in S$.
- (M4) If $\beta \in S$ and $\alpha \in C_\beta$, then $G_\alpha \subseteq G_\beta$.
- (M5) If for some β , $G_\beta(a) = b$ and for some $w \in W$, $w(a) = b$, then there is some γ so that $w \subseteq G_\gamma$. Furthermore if β is the minimum ordinal so that $G_\beta(a) = b$ then $\gamma = \beta$ or $\beta \in C_\gamma$.

In order to construct the set $M = \bigcup_{\alpha < \omega_3} M_\alpha$ and the mappings G_α we define an oriented graph with M as the set of vertices. We use the terminology of Serre [16] for graph-theoretic notions. If x is an edge, the origin of x is denoted by $o(x)$ and the terminus by $t(x)$. Our graph has an inverse edge \bar{x} for each edge x . Thus $o(\bar{x}) = t(x)$ and $t(\bar{x}) = o(x)$. Some edges are called *positive*, the rest are called *negative*. An edge is positive if and only if its inverse is negative. For each edge x of M there is a set $L(x)$ of labels. The set of possible labels for positive edges is $\{g_\alpha : \alpha < \omega_3\}$. The negative edges can have elements of $\{g_\alpha^{-1} : \alpha < \omega_3\}$ as labels. The labels are assumed to be given in such a way that a positive edge gets g_α as a label if and only if its inverse gets

the label g_α^{-1} . During the construction the sets of labels will be extended step by step.

The construction is analogous to building an acyclic graph on which a group acts freely. The graph then turns out to be the Cayley graph of the group. The labelled graph we will build will be the “Cayley graph” of W which will be as free as possible given (M1)–(M4). Condition (M5) is a consequence of the freeness of the construction.

Let us suppose the sets M_β , $\beta < \alpha$, of vertices have been defined. Let $M_{<\alpha} = \bigcup_{\beta < \alpha} M_\beta$. Some vertices in $M_{<\alpha}$ have edges between them and a set $L(x)$ of labels has been assigned to each such edge x .

If α is a limit ordinal, we let $M_\alpha = M_{<\alpha}$. So let us assume $\alpha = \beta + 1$. If $\beta \notin S$, $M_\alpha = M_\beta$. So let us assume $\beta \in S$. Let $\gamma = \sup(C_\beta)$. Notice that since S consists entirely of limit ordinals and $C_\beta \subseteq S$, either $\gamma = \beta$ or $\gamma + 1 < \beta$.

Case 1. $\gamma = \beta$: We extend M_β to M_α by adding new vertices $\{P_z : z \in \mathbb{Z}\}$ and for each $z \in \mathbb{Z}$ a positive edge $x_\alpha^{P_z}$ with $o(x_\alpha^{P_z}) = P_z$ and $t(x_\alpha^{P_z}) = P_{z+1}$. We also let $L(x_\alpha^{P_z}) = \{g_\beta\} \cup \{g_\delta : \beta \in C_\delta\}$.

Case 2. $\gamma + 1 < \beta$: We extend M_β to M_α by adding new vertices $\{P'_z : z \in \mathbb{Z} \setminus \{0\}\}$ for each $P \in M_\beta \setminus M_{\gamma+1}$. For notational convenience let $P'_0 = P$. Now we add for each $P \in M_\beta \setminus M_{\gamma+1}$ new edges as follows. For each $z \in \mathbb{Z}$ we add a positive edge $x_\alpha^{P'_z}$ with

$$o(x_\alpha^{P'_z}) = P'_z, \quad t(x_\alpha^{P'_z}) = P'_{z+1}, \quad L(x_\alpha^{P'_z}) = \{g_\beta\} \cup \{g_\delta : \beta \in C_\delta\}.$$

This determines completely the inverse of $x_\alpha^{P'_z}$.

This ends the construction of the graph. In the construction each vertex P in $M_{\alpha+1}$, $\alpha \in S$, is made the origin of a unique edge x_α^P with $g_\alpha \in L(x_\alpha^P)$. We define $G_\alpha(P) = t(x_\alpha^P)$.

The construction of the sets M_α and the mappings G_α is now completed. It follows immediately from the construction that each G_α , $\alpha \in S$, is a bijection of $M_{\alpha+1}$. So (M1)–(M3) hold. (M4) holds, because g_α is added to the labels of any edge with g_β , where $\beta \in C_\alpha$, as a label. Finally, (M5) is a consequence of the fact that the graph is circuit-free.

Let us fix $a_0 \in M_1$ and $b_0 = G_{\beta_0}(a_0)$, where $\beta_0 \in C_\alpha$ for all $\alpha \in S$. Note that we may assume, without loss of generality, the existence of such a β_0 .

If $a_0, a_1 \in M$, let

$$R_{(a_0, a_1)} = \{(a'_0, a'_1) \in M^2 : \exists w \in W (w(a_0) = a'_0 \wedge w(a_1) = a'_1)\}.$$

We let

$$\mathfrak{M} = \langle M, (R_{(a_0, a_1)})_{(a_0, a_1) \in M^2} \rangle, \quad \mathfrak{A} = \langle \mathfrak{M}, a_0 \rangle, \quad \mathfrak{B} = \langle \mathfrak{M}, b_0 \rangle,$$

and show that $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is nondetermined.

The reduction of the language of \mathfrak{A} and \mathfrak{B} to one binary predicate is easy. One just adds a copy of ω_3 , together with its ordering, and a copy of $M \times M$ to the structures with the projection maps. Then fix a bijection ϕ from ω_3 to M^2 .

Add a new binary predicate R to the language and interpret R to be contained in $\omega_3 \times M^2$ such that $R(\beta, (a, b))$ holds if and only if $R_{\phi(\beta)}(a, b)$ holds. We can now dispense with the old binary predicates. We have replaced our structure by one in a finite language without making any difference to who wins the game $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$. The extra step of reducing to a single binary predicate is standard.

An important property of these models is that if $\alpha \in S \cap S_1^3$, then $G_\alpha \upharpoonright M_\alpha$ is an automorphism of the restriction of \mathfrak{M} to M_α and takes a_0 to b_0 .

Claim 3. \forall does not win $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$.

Suppose \forall has a winning strategy τ . Again, there is a quick argument which uses CH: Find $\alpha \in S$ such that M_α is closed under τ and $\text{cf}(\alpha) = \omega_1$. Now C_α is cub on α , whence G_α maps M_α onto itself. Using G_α player \exists can easily beat τ , a contradiction.

In the following longer argument we need not assume CH. Let κ be a large regular cardinal. Let \mathfrak{H} be the expansion of $\langle H(\kappa), \in \rangle$ obtained by adding the following structure to it:

- (H1) The function $\alpha \mapsto M_\alpha$.
- (H2) The function $\alpha \mapsto G_\alpha$.
- (H3) The function $\alpha \mapsto C_\alpha$.
- (H4) A well-ordering $<^*$ of the universe.
- (H5) The winning strategy τ .
- (H6) The sets S and T .

Let $\mathfrak{N} = \langle N, \in, \dots \rangle$ be an elementary submodel of \mathfrak{H} of cardinality \aleph_2 such that $\alpha = N \cap \omega_3 \in S \cap S_1^3$.

Now C_α is a cub of order-type ω_1 on α and G_α maps M_α onto M_α . Moreover, G_α is a partial isomorphism from \mathfrak{A} into \mathfrak{B} . Provided that τ does not lead \forall to play his moves outside M_α , \exists has an obvious strategy: he lets G_α determine his moves. So let us assume a sequence $\langle (x_\xi, y_\xi) : \xi < \gamma \rangle$ has been played inside M_α and $\gamma < \omega_1$. Let $\beta \in C_\alpha$ such that M_β contains the elements x_ξ, y_ξ for $\xi < \gamma$. The sequence $\langle y_\xi : \xi < \gamma \rangle$ is totally determined by G_β and τ . Since $G_\beta \in N$, $\langle y_\xi : \xi < \gamma \rangle \in N$, and we are done.

Claim 4. \exists does not win $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$.

Suppose \exists has a winning strategy τ . Let \mathfrak{H} be defined as above and $\mathfrak{N} = \langle N, \in, \dots \rangle$ be an elementary submodel of \mathfrak{H} of cardinality \aleph_2 such that $\alpha = N \cap \omega_3 \in T$. We let \forall play during the first ω moves of $\mathcal{G}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ a sequence $\langle a_n : n < \omega \rangle$ in \mathfrak{A} such that if α_n is the least α_n with $a_n \in M_{\alpha_n}$, then the sequence $\langle \alpha_n : n < \omega \rangle$ is ascending and $\sup\{\alpha_n : n < \omega\} = \alpha$. Let \exists respond following τ with $\langle b_n : n < \omega \rangle$. As his move number ω player \forall plays some element $a_\omega \in M \setminus M_\alpha$ in \mathfrak{A} and \exists answers according to τ with b_ω .

For all $i \leq \omega$, $R_{(a_0, a_i)}(a_0, a_i)$ holds. Hence $R_{(a_0, a_i)}(b_0, b_i)$ holds. So there is w_i such that $w_i(a_0) = b_0$ and $w_i(a_i) = b_i$. Since $G_{\beta_0}(a_0) = b_0$, by (M5), for each i there is β_i so that $G_{\beta_i}(a_i) = b_i$. We can assume that β_i is chosen to be minimal. Notice that for all i , $\beta_i > \alpha_i$ and for $i < \omega$, $\beta_i \in \mathfrak{N}$. So $\sup\{\beta_i : i < \omega\} = \alpha$.

Also, by the same reasoning as above, for each $i < \omega$, $R_{(a_i, a_\omega)}(b_i, b_\omega)$ holds. Applying (M5), we get that $G_{\beta_\omega}(a_i) = b_i$. Using (M5) again and the minimality of β_i , for all $i < \omega$, $\beta_i \in C_{\beta_\omega}$. Thus α is a limit of elements of C_{β_ω} , contradicting $\alpha \in T$. \square

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